The Prop determines $\omega_\alpha{}^\beta, \tau^\beta$ uniquely.

We define a connection on $\mathcal{V}$

$$\nabla Z_\alpha = \omega_\alpha{}^\beta Z_\beta ,$$

i.e. ∇ is a linear map

sections of $\mathcal{V}$ $\longrightarrow$ $\mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{V}) \otimes \Lambda^1(M)$ ← 1-forms

that satisfies Leibnitz rule $\nabla(uZ) = du \otimes Z + u \nabla Z$.

Def. • ∇ is the Tanaka-Webster connection associated with the contact form θ.

• τ^α is the Tanaka-Webster torsion.

• $(M, \mathcal{V}, \theta, \nabla)$ is a pseudohermitian structure.

• ∇ acts on any tensor bundle on $T^{1,0}, T^{0,1}$ + duals.

• $\omega_\alpha{}^\beta$ are the connection forms, and

$$\begin{cases} d\theta = i g_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta} \\ d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + \theta \wedge \tau^\alpha \\ dg_{\alpha\bar\beta} = \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha}, \quad \tau^\alpha = A_{\bar\mu}{}^\alpha \theta^{\bar\mu}, \quad A_{\alpha\beta} = A_{\beta\alpha} \end{cases}$$

The curvature. The curvature form of the connection ∇ is given by

$$\Pi_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta.$$

Prop (Webster)

$$\Pi_\alpha{}^\beta = R_\alpha{}^\beta{}_{\nu\bar\mu}\,\theta^\nu \wedge \theta^{\bar\mu} + W_\alpha{}^\beta{}_\gamma\,\theta^\gamma \wedge \theta - W^\beta{}_{\alpha\bar\gamma}\,\theta^{\bar\gamma} \wedge \theta$$

$$+\, i\,\theta_\alpha \wedge \tau^\beta - i\,\tau_\alpha \wedge \theta^\beta,$$

where $R_{\alpha\bar\beta\nu\bar\mu}$ has Hermitian curvature symmetries and $W_\alpha{}^\beta{}_\gamma = \nabla^\beta A_{\alpha\gamma}$.

Pf. See [Webster]. Diff. previously established structure eq's. ☒

Def $R_{\alpha\bar\beta\nu\bar\mu}$ is the Tanaka-Webster pseudo hermitian curvature.

Recall that any other admissible coframe (preserving the Levi form $g_{\alpha\bar\beta}$) is given by

$$\hat\theta^\alpha = t_\beta{}^\alpha\,\theta^\beta\,,\quad (t_\beta{}^\alpha)\in U(n,\mathbb{C})$$

Ψ-Hermitian curvature and torsion transform as tensors

$$\begin{cases} R_{\alpha\bar\beta\nu\bar\mu} = \hat R_{\gamma\bar\varepsilon\delta\bar\varkappa}\, t_\alpha{}^\gamma\, \overline{t_\beta{}^\varepsilon}\, t_\nu{}^\delta\, \overline{t_\mu{}^\varkappa} \\ A_{\alpha\beta} = A_{\nu\mu}\, t_\alpha{}^\nu\, t_\beta{}^\mu \end{cases}$$

Representation theory $\Rightarrow$ a Hermitian curvature tensor R decomposes under action of $U(n,\mathbb{C})$ as

$$R = B \oplus \mathring{Ric} \mathbin{\bigcirc\mkern-15mu\wedge} g \oplus R \mathbin{\bigcirc\mkern-15mu\wedge} g^2$$

Scalar curvature component

Bochner tensor

traceless Ricci component

Ricci: $R_{\alpha\bar\beta} = R_{\gamma}{}^{\gamma}{}_{\alpha\bar\beta}$

Scalar $R = R_{\gamma}{}^{\gamma}$

Traceless Ricci: $\mathring{R}_{\alpha\bar\beta} = R_{\alpha\bar\beta} - \frac{1}{n} R g_{\alpha\bar\beta}$

Bochner:

$$B_{\alpha\bar\beta\nu\bar\mu} = R_{\alpha\bar\beta\nu\bar\mu} - \frac{1}{n+2}\left(R_{\alpha\bar\beta} g_{\nu\bar\mu} + R_{\alpha\bar\mu} g_{\nu\bar\beta} + R_{\nu\bar\beta} g_{\alpha\bar\mu} + R_{\nu\bar\mu} g_{\alpha\bar\beta}\right) + \frac{R}{(n+1)(n+2)}\left(g_{\alpha\bar\beta} g_{\nu\bar\mu} + g_{\alpha\bar\mu} g_{\nu\bar\beta}\right).$$

Thm. (Chern, Moser, Webster)

Bochner $\cong$ CR curvature

From Moser normal form introduced earlier.

Thus, $B_{\alpha\bar{\beta}\nu\bar{\mu}} \cong S_{\alpha\bar{\beta}\nu\bar{\mu}}$ and we recall

Thm $(M,\mathcal{V})$ is spherical at $p \in M$ (locally CR diffeomorphic to $S^{2n+1} \subseteq \mathbb{C}^{n+1}$) $\overset{2}{\Longleftrightarrow}$ $S_{\alpha\bar{\beta}\nu\bar{\mu}} = 0$ $\forall q$ in an open nbhd of p.

An important problem is the context of s. ψcvx CR mflds (M, T).

We introduce the $\overline{\partial}_b$-operator. It is associated to the CR structure but can be introduced in the context of a ψ-herm. structure. So let θ be a contact form, $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ an admissible coframe w/ dual frame $(T, Z_\alpha, Z_{\bar\alpha})$. Decompose the connection form

$$\omega_\alpha{}^\beta = \omega_\alpha{}^\beta{}_\gamma \theta^\gamma + \omega_\alpha{}^\beta{}_{\bar\mu} \theta^{\bar\mu} + \omega_\alpha{}^\beta{}_0 \theta$$

The connection ∇ decomposes accordingly

$$\nabla_\alpha Z_\beta = \omega_\alpha{}^\gamma{}_\beta Z_\gamma \quad \text{etc.}$$

A $(0,q)$-form on M is a form of the type

$$\Omega = \Omega_{\bar\alpha_1 \dots \bar\alpha_q} \theta^{\bar\alpha_1} \wedge \dots \wedge \theta^{\bar\alpha_q}$$

where $\Omega_{\bar\alpha_1 \dots \bar\alpha_q}$ is a tensor field w/ appropriate skew symmetries

We define the $\bar\partial_b$ operator on $(0,q)$-forms

$$C^\infty = \Lambda^{0,0} \xrightarrow[\bar\partial_b]{} \Lambda^{0,1} \xrightarrow[\bar\partial_b]{} \cdots \xrightarrow[\bar\partial_b]{} \Lambda^{0,n}$$

by $\bar\partial_b u = \nabla_{\bar\alpha} u \,\theta^{\bar\alpha}$, $\bar\partial_b (f_{\bar\beta} \theta^{\bar\beta}) = \nabla_{\bar\alpha} f_{\bar\beta} \theta^{\bar\alpha} \wedge \theta^{\bar\beta}$.

etc. It is convenient to consider $(0,q)$-forms as tensor fields $\Rightarrow$

$$\bar\partial_b f_{\bar\beta} = \nabla_{\bar\alpha} f_{\bar\beta} - \nabla_{\bar\beta} f_{\bar\alpha} \quad \text{etc.}$$

<u>Problem</u>. Develop theory of $\bar\partial_b$ on domains $G \subseteq M$ including estimates in suitable Sobolev (Folland-Stein) spaces, parallel to the theory on φcvx domains G in $\mathbb{C}^N$.

Local

<u>Application</u>. CR embedding problem: Given $(M,\mathcal{D})$ s. φcvx, CR dim $M = n$. For $p_0 \in M$, can we embed $(M,\mathcal{D})$ near p_0 as a real hypersurface in $\mathbb{C}^{n+1}$?

Thm.

Kuranishi: Yes if $n \geq 9$.

Akahori: Yes if $n = 7$.

Nirenberg: No if $n = 3$.

Open $n = 5$?